

Two integrable differential-difference equations derived from NLS-type equation

Zong-Wei Xu[†], Guo-Fu Yu[†] and Yik-Man Chiang[‡],

[†]Department of Mathematics, Shanghai Jiao Tong University,
Shanghai 200240, P.R. China

[‡] Department of Mathematics, Hong Kong University of Science and Technology,
Clear Water Bay, Kowloon, Hong Kong, P. R. China

Abstract

Two integrable differential-difference equations are derived from a (2+1)-dimensional modified Heisenberg ferromagnetic equation and a resonant nonlinear Schrödinger equation respectively. Multi-soliton solutions of the resulted semi-discrete systems are given through Hirota's bilinear method. Elastic and inelastic interaction behavior between two solitons are studied through the asymptotic analysis. Dynamics of two-soliton solutions are shown with graphs.

keywords: modified Heisenberg ferromagnetic system; Resonant nonlinear Schrödinger equation; Integrable discretization; Soliton interactions.

Mathematics Subject Classification (2000). 35Q53, 37K10, 35C05, 37K40

1 Introduction

Recently, integrable discretizations of integrable equations have been of considerable and current interest in soliton theory. As Suris mentioned [1], various approaches to the problem of integrable discretization are currently developed, among which the Hirota's bilinear method is very powerful and effective. Discrete analogues of almost all interesting soliton equations, the KdV, the Toda chain, the sine-Gordon, etc., can be obtained by the Hirota method. The purpose of this paper is to consider integrable discrete analogues of two nonlinear Schrödinger (NLS)-type system by Hirota method.

The one-dimensional classical continuum Heisenberg models with different magnetic interactions have been settled as one of the interesting and attractive classes of nonlinear dynamical equations exhibiting the complete integrability on many occasions. As is well known, Heisenberg first proposed in 1928 the following discrete (isotropic) Heisenberg ferromagnetic (DHF) spin chain [2]

$$\dot{\mathbf{S}}_n = \mathbf{S}_n \times (\mathbf{S}_{n+1} + \mathbf{S}_{n-1}), \quad (1)$$

where $\mathbf{S}_n = (s_1^n, s_2^n, s_3^n) \in \mathbf{R}^3$ with $|\mathbf{S}_n| = 1$ and the overdot represents the time derivative with respect to t . The DHF chain plays an important role in the theory of magnetism.

A performance of the standard continuous limit procedure leads DHF model (1) to the integrable Heisenberg ferromagnetic model

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \quad (2)$$

which is an important equation in condensed matter physics [3]. NLS-type equations are extensively used to describe nonlinear water waves in fluids, ion-acoustic waves in plasmas, nonlinear envelope pulses in the fibers. It is known that HF is gauge equivalent to NLS equation and DHF is gauge equivalent to a kind of discrete NLS-like equation [4].

Higher dimensional nonlinear evolution equations are proposed to describe certain nonlinear phenomena. Due to the dependence on the additional spatial variables in higher dimensional systems, richer solution structure might appear, such as dromions, lumps, breathers and loop solitons.

An extension of Eq. (2) is the $(2+1)$ -dimensional integrable modified HF system [5–8], as follows,

$$u_t + u_{xy} + uw = 0, \quad (3a)$$

$$v_t - v_{xy} - vw = 0, \quad (3b)$$

$$w_x + (uv)_y = 0, \quad (3c)$$

which is associated with the $(2+1)$ -dimensional NLS equation

$$iq_\tau + q_{\xi\tau} - 2q \int (|q|^2 q)_\eta d\xi = 0, \quad (4)$$

where η is a spatial variable y . System (3) can also be used to model the biological pattern formation in reaction-diffusion process [8, 9]. In Ref. [6], system (3) was investigated through the prolongation structure and Lax representation. In Refs. [7, 8], integrable property of system (3) was studied through the Painlevé analysis, and some localized coherent and periodic solutions were given by means of the multi-linear variable separation approach. Multi-soliton solutions of system (3) was derived in [10] by means of the Hirota bilinear method, and the double Wronskian solutions was given therein. Similar as the counterpart between DHF (1) and continuous HF (2), it is natural to consider discrete version of the $(2+1)$ -dimensional modified HF system (3).

A new integrable version of the nonlinear Schrödinger (NLS) equation, called the *resonant* NLS (RNLS) equation,

$$iU_t + U_{xx} + \frac{\alpha}{4}|U|^2 U = \beta \frac{1}{|U|} |U|_{xx} U, \quad (5)$$

was recently proposed [11] to describe low-dimensional gravity (the Jackiw-Teitelboim model) and response of a medium to the action of a quasimonochromatic wave. Here α is a nonlinear coefficient, and β denotes the strength of electrostriction pressure or diffraction. The term $|U|_{xx}/|U|$ on the righthand side of Eq. (5) is so called the "quantum potential". Moreover, Eq. (5) can model propagation of one-dimensional long magnetoacoustic waves in a cold collisionless plasma subject to a transverse magnetic field [12]. Note that when $\beta < 1$ Eq. (5) is reduced to NLS

$$i\Phi_\xi + \Phi_{\tau\tau} + \sigma|\Phi|^2\Phi = 0. \quad (6)$$

When $\beta > 1$, it is not reducible to the usual NLS equation but to a reaction-diffusion (RD) system [12–14].

The Lax pair in the 2×2 matrix form of Eq.(5) was given in [15]. Based on the RD version of RNLS, multi-soliton solutions were derived via the Bäcklund-Darboux transformations in [12] and via the Hirota method in [13] respectively. In [14], integrable extension of Eq.(5) has been suggested and soliton solutions of the resonant NLS case under the reduction condition have been presented via the Hirota method. Additionally, the method of binary Bell polynomials was used to study two-soliton solution and integrable properties of Eq.(5) in [16].

Since NLS equation has integrable discrete versions and RNLS is an intermediate equation between the focusing and defocusing NLS equation, it is natural to study the discrete analogue of RNLS. This paper is to investigate the integrable discretization of RNLS equation (5) in the case of $\beta > 1$.

The paper is structured as follows. In section 2, we present multi-soliton solutions and discrete version of $(2+1)$ -dimensional modified HF equation. In section 3, we study RNLS equation, including multi-soliton solutions, semi-discrete analogue and dynamic properties of solutions. Finally, a short conclusion is given in section 4.

2 $(2+1)$ -dimensional modified HF equation

2.1 N -soliton solution to modified HF system

Through the variable transformation

$$u = \frac{G}{F}, \quad v = \frac{H}{F}, \quad w = 2(\ln(F))_{xy}, \quad (7)$$

Eq.(3) transforms into the following bilinear form

$$(D_t + D_x D_y)G \bullet F = 0, \quad (8a)$$

$$(D_t - D_x D_y)H \bullet F = 0, \quad (8b)$$

$$D_x^2 F \bullet F + GH = 0. \quad (8c)$$

The Hirota bilinear differential operator $D_x^m D_t^k$ is defined by [17]

$$D_x^m D_y^n a \bullet b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^n a(x, y) b(x', y')|_{x'=x, y'=y},$$

$$m, n = 0, 1, 2, \dots$$

One-, two- and three-soliton solutions of bilinear equation set (8) are given in [10]. Note that the bilinear equations (8) are in Schrödinger type, here we present a compact form of multi-soliton solutions to the system (8). The two-soliton is given as

$$F = 1 + a(1, 1^*) \exp(\eta_1 + \xi_1) + a(1, 2^*) \exp(\eta_1 + \xi_2) \\ + a(2, 1^*) \exp(\eta_2 + \xi_1) + a(2, 2^*) \exp(\eta_2 + \xi_2) \\ + a(1, 2, 1^*, 2^*) \exp(\eta_1 + \eta_2 + \xi_1 + \xi_2), \quad (9)$$

$$G = \exp(\eta_1) + \exp(\eta_2) + a(1, 2, 1^*) \exp(\eta_1 + \eta_2 + \xi_1) \\ + a(1, 2, 2^*) \exp(\eta_1 + \eta_2 + \xi_2), \quad (10)$$

$$H = \exp(\xi_1) + \exp(\xi_2) + a(1, 1^*, 2^*) \exp(\eta_1 + \xi_1 + \xi_2) \\ + a(2, 1^*, 2^*) \exp(\eta_2 + \xi_1 + \xi_2). \quad (11)$$

with

$$\eta_i = k_i x + l_i y + \omega_i t + \eta_i^0, \quad w_i = -k_i l_i, \quad (12)$$

$$\xi_i = p_i x + q_i y + \Omega_i t + \xi_i^0, \quad \Omega_i = p_i q_i, \quad (13)$$

and the coefficients are defined by

$$a(i, j) = -2(k_i - k_j)^2, \quad (14)$$

$$a(i, j^*) = -\frac{1}{2(k_i + p_j)^2}, \quad (15)$$

$$a(i^*, j^*) = -2(p_i - p_j)^2, \quad (16)$$

$$a(i_1, i_2, \dots, i_n) = \prod_{1 \leq l < k \leq n} a(i_l, i_k). \quad (17)$$

Generally, exact N -soliton solution to Eq. (8) is expressed in the following form

$$F = \sum_{\mu=0,1}^{(e)} \exp \left[\sum_{j=1}^N \mu_j \eta_j + \sum_{j=N+1}^{2N} \mu_j \xi_{j-N} + \sum_{1 \leq i < j}^{2N} \mu_i \mu_j A_{ij} \right], \quad (18)$$

$$G = \sum_{\nu=0,1}^{(o)} \exp \left[\sum_{j=1}^N \nu_j \eta_j + \sum_{j=N+1}^{2N} \nu_j \xi_{j-N} + \sum_{1 \leq i < j}^{2N} \nu_i \nu_j A_{ij} \right], \quad (19)$$

$$H = \sum_{\lambda=0,1}^{(o)} \exp \left[\sum_{j=1}^N \lambda_j \eta_j + \sum_{j=N+1}^{2N} \lambda_j \xi_{j-N} + \sum_{1 \leq i < j}^{2N} \lambda_i \lambda_j A_{ij} \right], \quad (20)$$

with

$$\eta_j = k_j x + l_j y + \omega_j t, \quad \omega_j = -k_j l_j, \quad j = 1, 2, \dots, N, \quad (21)$$

$$\xi_j = p_j x + q_j y + \Omega_j t, \quad \Omega_j = p_j q_j, \quad j = 1, 2, \dots, N, \quad (22)$$

$$\exp(A_{ij}) = -2(k_i - k_j)^2, \quad i < j = 2, 3, \dots, N \quad (23)$$

$$\exp(A_{i,N+j}) = -\frac{1}{2(k_i + p_j)^2}, \quad i, j = 1, 2, \dots, N, \quad (24)$$

$$\exp(A_{N+i,N+j}) = -2(p_i - p_j)^2, \quad i < j = 2, 3, \dots, N. \quad (25)$$

Here α_j, γ_j are both real parameters relating respectively to the amplitude and phase of the i th soliton. The sum $\sum_{\mu=0,1}^{(e)}$ indicates the summation over all possible combinations of $\mu_i = 0, 1$ under the condition

$$\sum_{j=1}^N \mu_j = \sum_{j=1}^N \mu_{N+j}, \quad (26)$$

and $\sum_{\nu=0,1}^{(o)}, \sum_{\lambda=0,1}^{(o)}$ indicate the summation over all possible combinations of $\lambda_i = 0, 1, \nu_i = 0, 1$ under the condition

$$\sum_{j=1}^N \nu_j = \sum_{j=1}^N \nu_{N+j} + 1, \quad (27)$$

$$\sum_{j=1}^N \lambda_j = \sum_{j=1}^N \lambda_{N+j} - 1. \quad (28)$$

The proof of the N -soliton solution here is similar to that of the combined Schrödinger-mKdV equation in [18] and can be completed by induction. One can check the details therein.

2.2 Integrable semi-discrete analogue of modified HF equation

In this section, we construct the integrable discretization of HF equation (5) by using Hirota's discretization method [19]. The Hirota bilinear difference operator $\exp(\delta D_n)$ is defined as [17],

$$\exp(\delta D_n) a(n) \bullet b(n) \equiv \exp \left[\delta \left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'} \right) \right] a(n) b(n') \big|_{n'=n} = a(n + \delta) b(n - \delta).$$

The differential-difference HF system is obtained by discretizing the spacial part of the bilinear Eq. (8),

$$D_x G \bullet F \rightarrow \frac{1}{\epsilon} (G_{n+1} F_n - G_n F_{n+1}), \quad (29)$$

$$D_x^2 F \bullet F \rightarrow \frac{2}{\epsilon^2} (F_{n+1} F_{n-1} - F_n^2), \quad (30)$$

where $x = n\epsilon$, n being integers and ϵ a spatial-interval. Substituting (29)-(30) into Eq.(8) results to

$$D_t G_n \bullet F_n + \frac{1}{\epsilon} D_y (G_{n+1} \bullet F_n - G_n \bullet F_{n+1}) = 0, \quad (31)$$

$$D_t H_n \bullet F_n - \frac{1}{\epsilon} D_y (H_{n+1} \bullet F_n - H_n \bullet F_{n+1}) = 0, \quad (32)$$

$$\frac{2}{\epsilon^2} (F_{n+1} F_{n-1} - F_n^2) + G_n H_n = 0. \quad (33)$$

We demand that the discretized bilinear forms are invariant under the gauge transformation:

$$F_n \rightarrow F_n \exp(q_0 n), \quad (34)$$

$$G_n \rightarrow G_n \exp(q_0 n), \quad (35)$$

$$H_n \rightarrow H_n \exp(q_0 n). \quad (36)$$

Then we find a gauge invariant semi-discrete bilinear HF equations

$$(\epsilon D_t + 2D_y)G_{n+1} \bullet F_n + (\epsilon D_t - 2D_y)G_n \bullet F_{n+1} = 0, \quad (37)$$

$$(\epsilon D_t - 2D_y)H_{n+1} \bullet F_n + (\epsilon D_t + 2D_y)H_n \bullet F_{n+1} = 0, \quad (38)$$

$$\frac{2}{\epsilon^2} (F_{n+1}F_{n-1} - F_n^2) + G_n H_n = 0, \quad (39)$$

or equivalently in compact form

$$(\epsilon D_t \cosh \frac{D_n}{2} + 2D_y \sinh \frac{D_n}{2})G_n \bullet F_n = 0, \quad (40)$$

$$(\epsilon D_t \cosh \frac{D_n}{2} - 2D_y \sinh \frac{D_n}{2})H_n \bullet F_n = 0, \quad (41)$$

$$2(F_{n+1}F_{n-1} - F_n^2) + \epsilon^2 G_n H_n = 0. \quad (42)$$

Let

$$u_n = \frac{G_n}{F_n}, \quad v_n = \frac{H_n}{F_n}, \quad r_n = \ln \frac{F_{n+1}}{F_n}. \quad (43)$$

Then Eqs. (37)-(39) are transformed into ordinary nonlinear form

$$(u_{n+1} + u_n)_t + (u_{n+1} - u_n)r_{n,t} + \frac{2}{\epsilon} \left[(u_{n+1} - u_n)_y + (u_{n+1} + u_n)r_{n,y} \right] = 0, \quad (44)$$

$$(v_{n+1} + v_n)_t + (v_{n+1} - v_n)r_{n,t} - \frac{2}{\epsilon} \left[(v_{n+1} - v_n)_y + (v_{n+1} + v_n)r_{n,y} \right] = 0, \quad (45)$$

$$\frac{2}{\epsilon^2} (e^{r_n - r_{n-1}} - 1) + u_n v_n = 0. \quad (46)$$

When we take the continuum limit $\epsilon \rightarrow 0$, (119)-(129) reduce to (77)-(78) and

$$2(\ln F)_{xx} + uv = 0. \quad (47)$$

Differentiating Eq.(130) with respect to variable y , we get Eq. (3c). Thus we regard (119)-(129) as a semi-discrete version of the HF system (3). In the following discussion we take the interval $\epsilon = 1$ for the sake of simplicity.

Following the Hirota method, we expand G_n, H_n and F_n in series with a small parameter δ as

$$F_n = 1 + \delta^2 F_n^{(2)} + \delta^4 F_n^{(4)} + \dots + \delta^{2k} F_n^{(2k)} + \dots, \quad (48)$$

$$G_n = \delta G_n^{(1)} + \delta^3 G_n^{(3)} + \dots + \delta^{(2k+1)} G_n^{(2k+1)} + \dots, \quad (49)$$

$$H_n = \delta H_n^{(1)} + \delta^3 H_n^{(3)} + \dots + \delta^{(2k+1)} H_n^{(2k+1)} + \dots. \quad (50)$$

Substituting the expansion into the above bilinear Eqs. (37)-(39), we find that there are only odd order terms of δ in the first two equations while only even order terms appear in the third one. By the standard direct perturbation method, we obtain the one-soliton solution

$$G_n = \gamma_1 \exp(\eta_1), \quad H_n = \gamma'_1 \exp(\eta'_1), \quad F_n = 1 - \frac{\gamma_1 \gamma'_1 \beta_1 \beta'_1}{(\beta_1 \beta'_1 - 1)^2} \exp(\eta_1 + \eta'_1), \quad (51)$$

where $\eta_1 = p_1 t + q_1 y + \ln(\beta_1)n$, $\eta'_1 = p'_1 t + q'_1 y + \ln(\beta'_1)n$, and β_1, β'_1 satisfy

$$\beta_1 = \frac{2q_1 - p_1}{2q_1 + p_1}, \quad (52)$$

$$\beta'_1 = \frac{2q'_1 + p'_1}{2q'_1 - p'_1}, \quad (53)$$

α_1, γ_1 and α'_1, γ'_1 are arbitrary constants. The two-soliton solution is presented as

$$G_n = \delta [\exp(\eta_1) + \exp(\eta_2)] + \delta^3 [a_{121} \exp(\eta_1 + \eta_2 + \eta'_1) + a_{122} \exp(\eta_1 + \eta_2 + \eta'_2)], \quad (54)$$

$$H_n = \delta [\exp(\eta'_1) + \exp(\eta'_2)] + \delta^3 [b_{121} \exp(\eta'_1 + \eta'_2 + \eta_1) + b_{122} \exp(\eta'_1 + \eta'_2 + \eta_2)], \quad (55)$$

$$F_n = 1 - \delta^2 \left[\frac{\gamma_1 \gamma'_1 \beta_1 \beta'_1 \exp(\eta_1 + \eta'_1)}{2(\beta_1 \beta'_1 - 1)^2} + \frac{\gamma_1 \gamma'_2 \beta_1 \beta'_2 \exp(\eta_1 + \eta'_2)}{2(\beta_1 \beta'_2 - 1)^2} + \frac{\gamma_2 \gamma'_1 \beta_2 \beta'_1 \exp(\eta_2 + \eta'_1)}{2(\beta_2 \beta'_1 - 1)^2} \right. \\ \left. + \frac{\gamma_2 \gamma'_2 \beta_2 \beta'_2 \exp(\eta_2 + \eta'_2)}{2(\beta_2 \beta'_2 - 1)^2} \right] + \delta^4 \chi_{1212} [\exp(\eta_1 + \eta_2 + \eta'_1 + \eta'_2)] \quad (56)$$

where

$$\begin{aligned}
a_{121} &= -\frac{\gamma_1 \gamma_2 \gamma'_1 (\beta_1 - \beta_2)^2 \beta_1'^2}{2(\beta_2 \beta_1' - 1)^2 (\beta_1 \beta_1' - 1)^2}, & a_{122} &= -\frac{\gamma_1 \gamma_2 \gamma'_2 (\beta_1 - \beta_2)^2 \beta_2'^2}{2(\beta_2 \beta_2' - 1)^2 (\beta_1 \beta_2' - 1)^2}, \\
b_{121} &= -\frac{\gamma_1 \gamma'_1 \gamma'_2 (\beta_1' - \beta_2')^2 \beta_1^2}{2(\beta_1 \beta_1' - 1)^2 (\beta_1 \beta_2' - 1)^2}, & b_{122} &= -\frac{\gamma_2 \gamma'_1 \gamma'_2 (\beta_1' - \beta_2')^2 \beta_2^2}{2(\beta_2 \beta_2' - 1)^2 (\beta_2 \beta_1' - 1)^2}, \\
\chi_{1212} &= \frac{(\beta_1 - \beta_2)^2 (\beta_1' - \beta_2')^2 \gamma_1 \gamma_2 \gamma'_1 \gamma'_2 \beta_1 \beta_2 \beta_1' \beta_2'}{4(\beta_1 \beta_1' - 1)^2 (\beta_1 \beta_2' - 1)^2 (\beta_2 \beta_1' - 1)^2 (\beta_2 \beta_2' - 1)^2}.
\end{aligned} \tag{57}$$

We can use the following compact expression for the above two-soliton solution,

$$\begin{aligned}
F_n &= 1 + a(1, 1^*) \gamma_1 \gamma'_1 \exp(\eta_1 + \eta'_1) + a(1, 2^*) \gamma_1 \gamma'_2 \exp(\eta_1 + \eta'_2) \\
&\quad + a(2, 1^*) \gamma_2 \gamma'_1 \exp(\eta_2 + \eta'_1) + a(2, 2^*) \gamma_2 \gamma'_2 \exp(\eta_2 + \eta'_2) \\
&\quad + a(1, 2, 1^*, 2^*) \gamma_1 \gamma'_1 \gamma_2 \gamma'_2 \exp(\eta_1 + \eta_2 + \eta'_1 + \eta'_2),
\end{aligned} \tag{58}$$

$$\begin{aligned}
G_n &= \gamma_1 \exp(\eta_1) + \gamma_2 \exp(\eta_2) + a(1, 2, 1^*) \gamma_1 \gamma_2 \gamma'_1 \exp(\eta_1 + \eta_2 + \eta'_1) \\
&\quad + a(1, 2, 2^*) \gamma_1 \gamma_2 \gamma'_2 \exp(\eta_1 + \eta_2 + \eta'_2),
\end{aligned} \tag{59}$$

$$\begin{aligned}
H_n &= \gamma'_1 \exp(\eta'_1) + \gamma'_2 \exp(\eta'_2) + a(1, 1^*, 2^*) \gamma_1 \gamma'_1 \gamma'_2 \exp(\eta_1 + \eta'_1 + \eta'_2) \\
&\quad + a(2, 1^*, 2^*) \gamma_2 \gamma'_1 \gamma'_2 \exp(\eta_2 + \eta'_1 + \eta'_2),
\end{aligned} \tag{60}$$

where the coefficients are defined as

$$a(i, j) = -2 \frac{(\beta_i - \beta_j)^2}{\beta_i \beta_j}, \tag{61}$$

$$a(i, j^*) = -\frac{\beta_i \beta_j'}{(\beta_i \beta_j' - 1)^2}, \tag{62}$$

$$a(i^*, j^*) = -2 \frac{(\beta_i' - \beta_j')^2}{\beta_i' \beta_j'}, \tag{63}$$

and $a(i, j, k^*)$, $a(i, j^*, k^*)$, $a(i, j, k^*, l^*)$ satisfy the operation rule (17). In the same way, we can construct the three-soliton solution as that in [20]. The above expressions of the one- and two-soliton solutions suggest the exact N -soliton solution of Eqs. (37)-(39) in the following form

$$F_n = \sum_{\mu=0,1}^{(e)} \exp \left[\sum_{j=1}^N \mu_j \eta_j + \sum_{j=N+1}^{2N} \mu_j \eta'_{j-N} + \sum_{1 \leq k < l}^{2N} \mu_k \mu_l A_{kl} \right], \tag{64}$$

$$G_n = \sum_{\nu=0,1}^{(o)} \exp \left[\sum_{j=1}^N \nu_j \eta_j + \sum_{j=N+1}^{2N} \nu_j \eta'_{j-N} + \sum_{1 \leq k < l}^{2N} \nu_k \nu_l A_{kl} \right], \tag{65}$$

$$H_n = \sum_{\lambda=0,1}^{(o)} \exp \left[\sum_{j=1}^N \lambda_j \eta_j + \sum_{j=N+1}^{2N} \lambda_j \eta'_{j-N} + \sum_{1 \leq k < l}^{2N} \lambda_k \lambda_l A_{kl} \right], \tag{66}$$

where

$$\eta_j = p_j t + q_j y + \ln(\beta_j) n + \gamma_j, \quad \beta_j = \frac{2q_j - p_j}{2q_j + p_j}, \quad j = 1, 2, \dots, N, \tag{67}$$

$$\eta'_j = p'_j t + q'_j y + \ln(\beta'_j) n + \gamma'_j, \quad \beta'_j = \frac{2q'_j + p'_j}{2q'_j - p'_j}, \quad j = 1, 2, \dots, N, \tag{68}$$

$$\exp(A_{kl}) = -2 \frac{(\beta_k - \beta_l)^2}{\beta_k \beta_l}, \quad k < l = 2, 3, \dots, N \tag{69}$$

$$\exp(A_{k,N+l}) = -\frac{1}{2} \frac{\beta_k \beta'_l}{(\beta_k \beta'_l - 1)^2}, \quad k, l = 1, 2, \dots, N, \tag{70}$$

$$\exp(A_{N+k,N+l}) = -2 \frac{(\beta'_k - \beta'_l)^2}{\beta'_k \beta'_l}, \quad k < l = 2, 3, \dots, N. \tag{71}$$

Here $p_j, q_j, \beta_j, \gamma_j$ are all real parameters. The summations $\sum_{\mu=0,1}^{(e)}$, $\sum_{\nu=0,1}^{(o)}$ and $\sum_{\lambda=0,1}^{(o)}$ satisfy the condition (26)-(28) respectively.

2.3 Soliton propagation and interaction

By virtue of (43), (54)-(57), interaction between the two solitons can be investigated. From Fig. 1, head-on elastic interaction between the two solitons is found. With the time evolution, two solitons travel towards each other and then separate. After the interaction, the two solitons maintain their original amplitudes, velocities and shapes except for the phase shifts. To interpret the elastic interaction behavior of two solitons under the condition $\gamma_1\gamma_2\gamma'_1\gamma'_2 \neq 0$, asymptotic analysis is carried out in the following.

First we take the notation $\chi_{ij} = a(i, j^*)$ and let

$$A = \{|\eta_1 + \eta'_1|, |\eta_1 + \eta'_2|, |\eta_2 + \eta'_1|, |\eta_2 + \eta'_2|, |\eta_1 - \eta_2|, |\eta'_1 - \eta'_2|\}.$$

When $\gamma_1\gamma_2\gamma'_1\gamma'_2 \neq 0$, we have the following asymptotic behaviors as

$$\phi = \frac{GH}{F^2} \sim \begin{cases} \frac{\gamma_1\gamma'_1}{|\gamma_1\gamma'_1\chi_{11}|} \text{sech}^2\left(\frac{\eta_1+\eta'_1+\ln|\gamma_1\gamma'_1\chi_{11}|}{2}\right) & \eta_1 + \eta'_1 \text{ fixed}, \eta_2 + \eta'_2 \rightarrow -\infty, |\eta_2 + \eta'_2| = \max A \\ \frac{\gamma_1\gamma'_1}{|\gamma_1\gamma'_1\chi_{11}|} \text{sech}^2\left(\frac{\eta_1+\eta'_1+\ln|\gamma_1\gamma'_1\chi_{1212}\chi_{22}^{-1}|}{2}\right) & \eta_1 + \eta'_1 \text{ fixed}, \eta_2 + \eta'_2 \rightarrow +\infty, |\eta_2 + \eta'_2| = \max A \\ \frac{\gamma_2\gamma'_2}{|\gamma_2\gamma'_2\chi_{22}|} \text{sech}^2\left(\frac{\eta_2+\eta'_2+\ln|\gamma_2\gamma'_2\chi_{22}|}{2}\right) & \eta_2 + \eta'_2 \text{ fixed}, \eta_1 + \eta'_1 \rightarrow -\infty, |\eta_1 + \eta'_1| = \max A \\ \frac{\gamma_2\gamma'_2}{|\gamma_2\gamma'_2\chi_{22}|} \text{sech}^2\left(\frac{\eta_2+\eta'_2+\ln|\gamma_2\gamma'_2\chi_{1212}\chi_{11}^{-1}|}{2}\right) & \eta_2 + \eta'_2 \text{ fixed}, \eta_1 + \eta'_1 \rightarrow +\infty, |\eta_1 + \eta'_1| = \max A \\ \frac{\gamma_1\gamma'_2}{|\gamma_1\gamma'_2\chi_{12}|} \text{sech}^2\left(\frac{\eta_1+\eta'_2+\ln|\gamma_1\gamma'_2\chi_{12}|}{2}\right) & \eta_1 + \eta'_2 \text{ fixed}, \eta_2 + \eta'_1 \rightarrow -\infty, |\eta_2 + \eta'_1| = \max A \\ \frac{\gamma_1\gamma'_2}{|\gamma_1\gamma'_2\chi_{12}|} \text{sech}^2\left(\frac{\eta_1+\eta'_2+\ln|\gamma_1\gamma'_2\chi_{1212}\chi_{21}^{-1}|}{2}\right) & \eta_1 + \eta'_2 \text{ fixed}, \eta_2 + \eta'_1 \rightarrow +\infty, |\eta_2 + \eta'_1| = \max A \\ \frac{\gamma_2\gamma'_1}{|\gamma_2\gamma'_1\chi_{21}|} \text{sech}^2\left(\frac{\eta_2+\eta'_1+\ln|\gamma_2\gamma'_1\chi_{21}|}{2}\right) & \eta_2 + \eta'_1 \text{ fixed}, \eta_1 + \eta'_2 \rightarrow -\infty, |\eta_1 + \eta'_1| = \max A \\ \frac{\gamma_2\gamma'_1}{|\gamma_2\gamma'_1\chi_{21}|} \text{sech}^2\left(\frac{\eta_2+\eta'_1+\ln|\gamma_2\gamma'_1\chi_{1212}\chi_{12}^{-1}|}{2}\right) & \eta_2 + \eta'_1 \text{ fixed}, \eta_1 + \eta'_2 \rightarrow +\infty, |\eta_1 + \eta'_1| = \max A \\ \frac{\gamma_1\gamma_2a_{122}}{|\gamma_1\gamma_2\chi_{12}\chi_{22}|} \text{sech}^2\left(\frac{\eta_1-\eta_2+\ln|\gamma_1\chi_{12}\gamma_2^{-1}\chi_{22}^{-1}|}{2}\right) & \eta_1 - \eta_2 \text{ fixed}, \eta'_1 - \eta'_2 \rightarrow -\infty, |\eta'_1 - \eta'_2| = \max A \\ \frac{\gamma_1\gamma_2a_{121}}{|\gamma_1\gamma_2\chi_{11}\chi_{21}|} \text{sech}^2\left(\frac{\eta_1-\eta_2+\ln|\gamma_1\chi_{11}\gamma_2^{-1}\chi_{21}^{-1}|}{2}\right) & \eta_1 - \eta_2 \text{ fixed}, \eta'_1 - \eta'_2 \rightarrow +\infty, |\eta'_1 - \eta'_2| = \max A \\ \frac{\gamma'_1\gamma'_2b_{212}}{|\gamma'_1\gamma'_2\chi_{21}\chi_{22}|} \text{sech}^2\left(\frac{\eta'_1-\eta'_2+\ln|\gamma'_1\chi_{21}\gamma'_2^{-1}\chi_{22}^{-1}|}{2}\right) & \eta'_1 - \eta'_2 \text{ fixed}, \eta_1 - \eta_2 \rightarrow -\infty, |\eta_1 + \eta'_1| = \max A \\ \frac{\gamma'_1\gamma'_2b_{112}}{|\gamma'_1\gamma'_2\chi_{11}\chi_{12}|} \text{sech}^2\left(\frac{\eta'_1-\eta'_2+\ln|\gamma'_1\chi_{11}\gamma'_2^{-1}\chi_{12}^{-1}|}{2}\right) & \eta'_1 - \eta'_2 \text{ fixed}, \eta_1 - \eta_2 \rightarrow +\infty, |\eta_1 + \eta'_1| = \max A \end{cases} \quad (72)$$

Plots of two solitons interaction at different times are shown in Fig. 1.

In addition, inelastic interaction between the two solitons can be found. When γ_i or γ'_i in (58)-(60) is chosen to be zero, soliton fusion and fission may occur during the interaction. Specially, when $\gamma'_2 = 0$ and

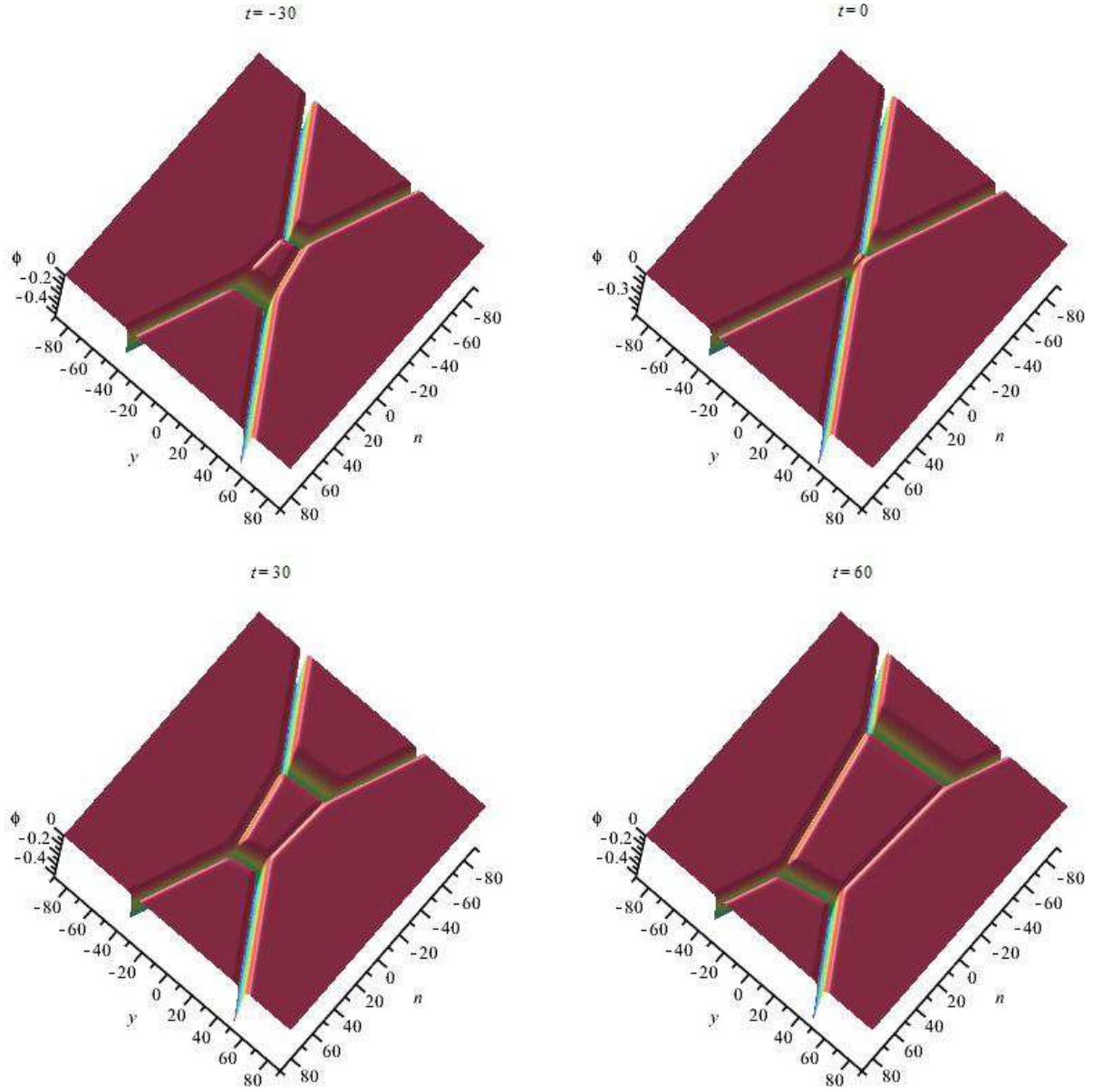


Fig. 1: Two-soliton interaction for $\phi = GH/F^2$ by (58)-(60) with parameters $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma'_1 = -1$, $\gamma'_2 = -1$, $p_1 = 0.2$, $p_2 = 0.3$, $q_1 = -0.7$, $q_2 = 0.6$, $p'_1 = 0.3$, $p'_2 = 0.2$, $q'_1 = -0.8$, $q'_2 = 0.75$

$\gamma_1\gamma_2\gamma'_1 \neq 0$, we have the following asymptotic analysis for ϕ ,

$$\phi = \frac{GH}{F^2} = \frac{\gamma_1\gamma'_1 e^{-(\eta_2+\eta'_2)} + \gamma_2\gamma'_1 e^{-(\eta_1+\eta'_2)} + \gamma_1\gamma_2\gamma'^2_1 a_{121} e^{\eta'_1-\eta'_2}}{\left(e^{-\frac{\eta_1+\eta'_1+\eta_2+\eta'_2}{2}} + \chi_{11}\gamma_1\gamma'_1 e^{\frac{\eta_1+\eta'_1-\eta_2-\eta'_2}{2}} + \chi_{21}\gamma_2\gamma'_1 e^{\frac{\eta_2+\eta'_1-\eta_1-\eta'_2}{2}} \right)^2}$$

$$\sim \begin{cases} \frac{\gamma_1\gamma'_1}{|\gamma_1\gamma'_1\chi_{11}|} \operatorname{sech}^2\left(\frac{\eta_1+\eta'_1+\ln|\gamma_1\gamma'_1\chi_{11}|}{2}\right) & \eta_1 + \eta'_1 \text{ fixed, } \eta_2 + \eta'_2 \rightarrow -\infty \\ \frac{\gamma_2\gamma'_1}{|\gamma_2\gamma'_1\chi_{21}|} \operatorname{sech}^2\left(\frac{\eta_2+\eta'_1+\ln|\gamma_2\gamma'_1\chi_{21}|}{2}\right) & \eta_2 + \eta'_1 \text{ fixed, } \eta_1 + \eta'_2 \rightarrow -\infty \\ \frac{\gamma_1\gamma_2 a_{121}}{|\gamma_1\gamma_2\chi_{11}\chi_{21}|} \operatorname{sech}^2\left(\frac{\eta_1-\eta_2+\ln|\gamma_1\chi_{11}\gamma_2^{-1}\chi_{21}^{-1}|}{2}\right) & \eta_1 - \eta_2 \text{ fixed, } \eta'_1 - \eta'_2 \rightarrow +\infty \end{cases}$$

Fig. 2 depicts one soliton split into two solitons with time evolution under the condition $\gamma'_2 = 0$.

3 Resonant nonlinear Schrödinger equation

3.1 multi-soliton solutions of RNLS equation

In [16], only two-soliton solution of RNLS equation are given. In this section, we present N -soliton solution. We introduce the transformation

$$U = \exp \frac{u + iau + v - iav}{2}, \quad (73)$$

where $u = u(x, t)$ and $v = v(x, t)$ are both real functions, and a is a real constant. Eq. (5) is transformed into

$$2(i-a)u_t + (1-\beta-a^2+2ia)u_x^2 + 2(1-\beta+ia)u_{xx} + 2(i+a)v_t + (1-\beta-a^2-2ia)v_x^2 + 2(1-\beta-ia)v_{xx} + 2(1-\beta+a^2)u_xv_x + \alpha \exp(u+v) = 0. \quad (74)$$

Since all the functions and parameters in Eq. (74) are real, separating the real and imaginary parts of Eq. (74) gives as

$$2au_t - (1-\beta-a^2)u_x^2 - 2(1-\beta)u_{xx} - 2av_t - (1-\beta-a^2)v_x^2 - 2(1-\beta)v_{xx} - 2(1-\beta+a^2)u_xv_x - \alpha \exp(u+v) = 0, \quad (75)$$

$$u_t + au_x^2 + au_{xx} + v_t - av_x^2 - av_{xx} = 0. \quad (76)$$

In what follows, we set $\beta = 1 + a^2$. Adding and subtracting (75) and (76) yield, respectively, two similar equations,

$$2(1-a)[u_t + au_x^2 + au_{xx}] + 2(1+a)[v_t - av_x^2 - av_{xx}] + \alpha e^{u+v} = 0, \quad (77)$$

$$2(1+a)[u_t + au_x^2 + au_{xx}] + 2(1-a)[v_t - av_x^2 - av_{xx}] - \alpha e^{u+v} = 0. \quad (78)$$

It is easy to check that provided Eqs.(77)-(78) hold automatically, the following two equations are satisfied,

$$u_t + a(u_{xx} + u_x^2) - \frac{\alpha}{4a}e^{u+v} = 0, \quad (79)$$

$$v_t - a(v_{xx} + v_x^2) + \frac{\alpha}{4a}e^{u+v} = 0. \quad (80)$$

Through the dependent variable transformation

$$u = \ln \frac{g}{f}, \quad v = \ln \frac{h}{f}, \quad (81)$$

Eqs. (79)-(80) are transformed into

$$\frac{4a}{gf}[(D_t + aD_x^2)g \bullet f] - \frac{1}{f^2}(4a^2D_x^2f \bullet f + \alpha gh) = 0, \quad (82)$$

$$\frac{4a}{hf}[(D_t - aD_x^2)h \bullet f] + \frac{1}{f^2}(4a^2D_x^2f \bullet f + \alpha gh) = 0, \quad (83)$$

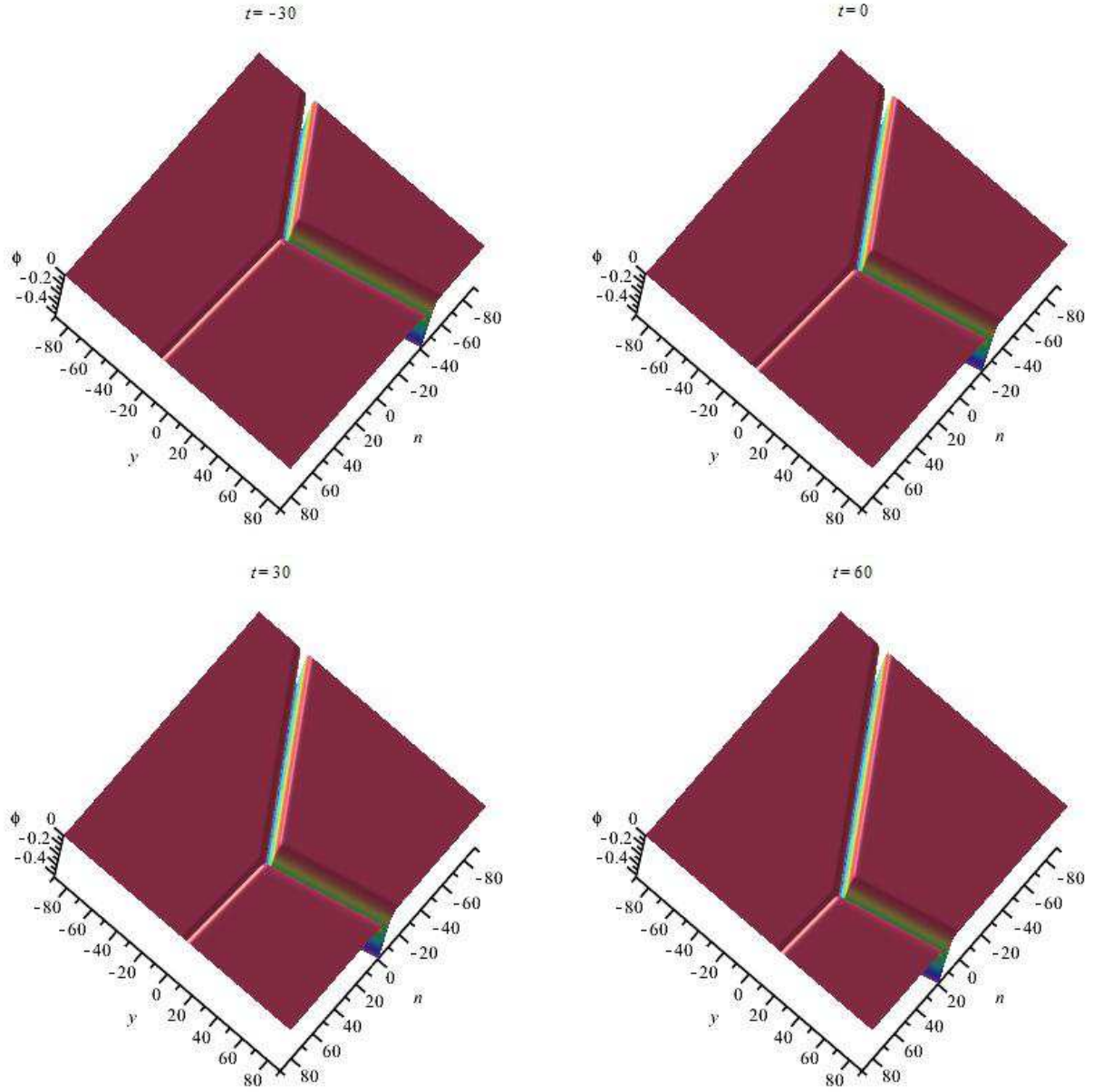


Fig. 2: One soliton splits into two solitons for $\phi = GH/F^2$ by (58)-(60) with parameters $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma'_1 = -1$, $\gamma'_2 = 0$, $p_1 = 0.2$, $p_2 = 0.3$, $q_1 = -0.7$, $q_2 = 0.6$, $p'_1 = 0.3$, $p'_2 = 0.2$, $q'_1 = -0.8$, $q'_2 = 0.75$.

that can be decoupled into the following bilinear equations

$$(D_t + aD_x^2)g \bullet f = 0, \quad (84a)$$

$$(D_t - aD_x^2)h \bullet f = 0, \quad (84b)$$

$$4a^2 D_x^2 f \bullet f + \alpha gh = 0. \quad (84c)$$

We remark here that the bilinear equations (84) are presented in [16] by using binary Bell polynomials method. To find multi-soliton solutions of Eqs. (84), we expand f, g and h as following

$$f = 1 + \epsilon^2 f^{(2)} + \epsilon^4 f^{(4)} + \dots, \quad (85)$$

$$g = \epsilon g^{(1)} + \epsilon^3 g^{(3)} + \dots, \quad (86)$$

$$h = \epsilon h^{(1)} + \epsilon h^{(3)} + \dots. \quad (87)$$

Substituting the expansion (85)-(87) into the bilinear Eq. (84) we find that there are only odd order terms of ϵ in the first two equations while only even order terms appear in the third one. Comparing the coefficients at each order of ϵ , we obtain one-soliton solution

$$g = \epsilon \exp(\eta), \quad h = \epsilon \exp(\xi), \quad (88)$$

$$f = 1 + \epsilon^2 \frac{-\alpha}{8a^2(p+r)^2} \exp(\eta + \xi), \quad (89)$$

where $\eta = px + qt + \eta^0, \xi = rx + st + \xi^0$ and q, s satisfy the dispersion relation

$$q = -ap^2, \quad s = ar^2. \quad (90)$$

Here p, r and η^0, ξ^0 are arbitrary constants. The two-soliton solution is presented as follows

$$g = \epsilon \left(\exp(\eta_1) + \exp(\eta_2) \right) + \epsilon^3 \left(a_{121} \exp(\eta_1 + \eta_2 + \xi_1) + a_{122} \exp(\eta_1 + \eta_2 + \xi_2) \right), \quad (91)$$

$$h = \epsilon \left(\exp(\xi_1) + \exp(\xi_2) \right) + \epsilon^3 \left(b_{121} \exp(\xi_1 + \xi_2 + \eta_1) + b_{122} \exp(\xi_1 + \xi_2 + \eta_2) \right), \quad (92)$$

$$f = 1 + \epsilon^2 \left[c_{11} \exp(\eta_1 + \xi_1) + c_{12} \exp(\eta_1 + \xi_2) + c_{21} \exp(\eta_2 + \xi_1) + c_{22} \exp(\eta_2 + \xi_2) \right] + \epsilon^4 c_{1212} \exp(\eta_1 + \eta_2 + \xi_1 + \xi_2), \quad (93)$$

where

$$\eta_i = p_i x + q_i t + \eta_i^0, \quad \xi_j = r_j x + s_j t + \xi_j^0, \quad (94)$$

$$q_i = -ap_i^2, \quad s_j = ar_j^2, \quad (95)$$

$$a_{121} = -\frac{\alpha(p_1 - p_2)^2}{8a^2(p_1 + r_1)^2(p_2 + r_1)^2}, \quad a_{122} = -\frac{\alpha(p_1 - p_2)^2}{8a^2(p_1 + r_2)^2(p_2 + r_2)^2}, \quad (96)$$

$$b_{121} = -\frac{\alpha(r_1 - r_2)^2}{8a^2(p_1 + r_1)^2(p_1 + r_2)^2}, \quad b_{122} = -\frac{\alpha(r_1 - r_2)^2}{8a^2(p_2 + r_1)^2(p_2 + r_2)^2}, \quad (97)$$

$$c_{ij} = -\frac{\alpha}{8a^2(p_i + k_j)^2}, \quad i, j = 1, 2, \quad (98)$$

$$c_{1212} = \frac{\alpha^2(p_1 - p_2)^2(r_1 - r_2)^2}{64a^4(p_1 + r_1)(p_1 + r_2)(p_2 + r_1)(p_2 + r_2)}. \quad (99)$$

We can use the following compact expression for the two-soliton solution,

$$g = \exp(\eta_1) + \exp(\eta_2) + a(1, 2, 1^*) \exp(\eta_1 + \eta_2 + \xi_1) + a(1, 2, 2^*) \exp(\eta_1 + \eta_2 + \xi_2), \quad (100)$$

$$h = \exp(\xi_1) + \exp(\xi_2) + a(1, 1^*, 2^*) \exp(\eta_1 + \xi_1 + \xi_2) + a(2, 1^*, 2^*) \exp(\eta_2 + \xi_1 + \xi_2), \quad (101)$$

$$f = 1 + a(1, 1^*) \exp(\eta_1 + \xi_1) + a(1, 2^*) \exp(\eta_1 + \xi_2) + a(2, 1^*) \exp(\eta_2 + \xi_1) + a(2, 2^*) \exp(\eta_2 + \xi_2) + a(1, 2, 1^*, 2^*) \exp(\eta_1 + \eta_2 + \xi_1 + \xi_2), \quad (102)$$

where the coefficients are defined as

$$a(i, j^*) = -\frac{\alpha}{8a^2(p_i + r_j)}, \quad (103)$$

$$a(i, j) = -\frac{8a^2(p_i - p_j)^2}{\alpha}, \quad (104)$$

$$a(i^*, j^*) = -\frac{8a^2(r_i - r_j)^2}{\alpha}, \quad (105)$$

$$a(i_1, i_2, \dots, i_n) = \prod_{1 \leq k < l \leq n} a(i_k, i_l). \quad (106)$$

In the same way, we give the exact N -soliton solution of Eqs. (84) in the following form

$$f = \sum_{\mu=0,1}^{(e)} \exp \left[\sum_{j=1}^N \mu_j \eta_j + \sum_{j=N+1}^{2N} \mu_j \xi_{j-N} + \sum_{1 \leq k < l}^{2N} \mu_k \mu_l A_{kl} \right], \quad (107)$$

$$g = \sum_{\nu=0,1}^{(o)} \exp \left[\sum_{j=1}^N \nu_j \eta_j + \sum_{j=N+1}^{2N} \nu_j \xi_{j-N} + \sum_{1 \leq k < l}^{2N} \nu_k \nu_l A_{kl} \right], \quad (108)$$

$$h = \sum_{\lambda=0,1}^{(o)} \exp \left[\sum_{j=1}^N \lambda_j \eta_j + \sum_{j=N+1}^{2N} \lambda_j \xi_{j-N} + \sum_{1 \leq k < l}^{2N} \lambda_k \lambda_l A_{kl} \right], \quad (109)$$

where

$$\eta_i = p_i x + q_i t + \eta_j^0, \quad q_j = -ap_j^2, \quad j = 1, 2, \dots, N, \quad (110)$$

$$\xi_j = r_j x + s_j t + \xi_j^0, \quad s_j = ar_j^2, \quad j = 1, 2, \dots, N, \quad (111)$$

$$\exp(A_{kl}) = -\frac{8a^2(p_k - p_l)^2}{\alpha}, \quad k < l = 2, 3, \dots, N \quad (112)$$

$$\exp(A_{k,N+l}) = -\frac{\alpha}{8a^2(p_k + r_l)}, \quad k, l = 1, 2, \dots, N, \quad (113)$$

$$\exp(A_{N+k,N+l}) = -\frac{8a^2(r_k - r_l)^2}{\alpha}, \quad k < l = 2, 3, \dots, N. \quad (114)$$

Here p_j, r_j are both real parameters relating respectively to the amplitude and phase of the i th soliton. The summations $\sum_{\mu=0,1}^{(e)}$, $\sum_{\nu=0,1}^{(o)}$ and $\sum_{\lambda=0,1}^{(o)}$ satisfy the condition (26)-(28) respectively.

3.2 Integrable discretization of RNLS equation

The differential-difference RNLS equation is obtained by discretizing the spacial part of the bilinear Eq. (84),

$$D_x^2 g \bullet f \rightarrow \frac{1}{\delta^2} (g_{n+1} f_{n-1} - 2g_n f_n + g_{n-1} f_{n+1}), \quad (115)$$

where $x = n\delta$, n being integers and δ a spatial-interval. We get

$$D_t g_n \bullet f_n + \frac{a}{\delta^2} (g_{n+1} f_{n-1} - 2g_n f_n + g_{n-1} f_{n+1}) = 0, \quad (116)$$

$$D_t h_n \bullet f_n - \frac{a}{\delta^2} (h_{n+1} f_{n-1} - 2h_n f_n + h_{n-1} f_{n+1}) = 0, \quad (117)$$

$$\frac{8a^2}{\delta^2} (f_{n+1} f_{n-1} - f_n^2) + \alpha g_n h_n = 0. \quad (118)$$

Let $g_n = e^{u_n} f_n$, $h_n = e^{v_n} f_n$. Then Eqs. (116)-(118) transforms into the following nonlinear form

$$u_{n,t} + \frac{a}{\delta^2} \left[\left(1 - \frac{\alpha \delta^2}{8a^2} e^{u_n + v_n} \right) (e^{u_{n+1} - u_n} + e^{u_{n-1} - u_n}) - 2 \right] = 0, \quad (119)$$

$$v_{n,t} - \frac{a}{\delta^2} \left[\left(1 - \frac{\alpha \delta^2}{8a^2} e^{u_n + v_n} \right) (e^{v_{n+1} - v_n} + e^{v_{n-1} - v_n}) + 2 \right] = 0. \quad (120)$$

When we take the continuum limit $\delta \rightarrow 0$, Eqs. (119)-(120) tend to Eqs. (79)-(80). In the sequel we use this discrete spacial step $\delta = 1$. Multiplying (119) by $2(1-a)(e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n})$, and (120) by $2(1+a)(e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n})$, adding and subtracting each other, respectively, yield

$$\begin{aligned} & 2(1+a) \left[u_{n,t} + a(e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n} - 2) \right] (e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n}) \\ & + 2(1-a) \left[v_{n,t} - a(e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n} + 2) \right] (e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n}) \\ & - \frac{\alpha}{2} e^{u_n+v_n} (e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n}) (e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n}) = 0, \end{aligned} \quad (121)$$

$$\begin{aligned} & 2(1-a) \left[u_{n,t} + a(e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n} - 2) \right] (e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n}) \\ & + 2(1+a) \left[v_{n,t} - a(e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n} + 2) \right] (e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n}) \\ & + \frac{\alpha}{2} e^{u_n+v_n} (e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n}) (e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n}) = 0. \end{aligned} \quad (122)$$

Adding and subtracting (121)-(122) give us

$$\begin{aligned} & \left[u_{n,t} + a(e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n} - 2) \right] (e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n}) \\ & + \left[v_{n,t} - a(e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n} + 2) \right] (e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n}) = 0, \end{aligned} \quad (123)$$

$$\begin{aligned} & 4a \left[u_{n,t} + a(e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n} - 2) \right] (e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n}) \\ & - 4a \left[v_{n,t} - a(e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n} + 2) \right] (e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n}) \\ & - \alpha e^{u_n+v_n} (e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n}) (e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n}) = 0. \end{aligned} \quad (124)$$

From (123) and (124), we get

$$\begin{aligned} & 4(i-a) \left[u_{n,t} + a(e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n} - 2) \right] (e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n}) \\ & + 4(a+i) \left[v_{n,t} - a(e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n} + 2) \right] (e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n}) \\ & + \alpha e^{u_n+v_n} (e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n}) (e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n}) = 0. \end{aligned} \quad (125)$$

Setting

$$U_n = \exp \left(\frac{u_n + v_n}{2} + ia \frac{u_n - v_n}{2} \right), \quad V_n = U_n^* = \exp \left(\frac{u_n + v_n}{2} - ia \frac{u_n - v_n}{2} \right), \quad (126)$$

or equivalently,

$$u_n = \ln(U_n^{\frac{a-i}{2a}} V_n^{\frac{a+i}{2a}}), \quad v_n = \ln(U_n^{\frac{a+i}{2a}} V_n^{\frac{a-i}{2a}}), \quad (127)$$

and substituting u_n, v_n by U_n, V_n into Eq. (125), we obtain a semi-discrete RNLS equation

$$\begin{aligned} & 4(i-a) \left[\operatorname{Re} \left(\frac{a-i}{a} \frac{U_{n,t}}{U_n} \right) + a \left(\left| \left(\frac{U_{n+1}}{U_n} \right)^{\frac{a-i}{2a}} \right|^2 + \left| \left(\frac{U_{n-1}}{U_n} \right)^{\frac{a-i}{2a}} \right|^2 - 2 \right) \right] \left[\left| \left(\frac{U_{n+1}}{U_n} \right)^{\frac{a+i}{2a}} \right|^2 + \left| \left(\frac{U_{n-1}}{U_n} \right)^{\frac{a+i}{2a}} \right|^2 \right] \\ & + 4(i+a) \left[\operatorname{Re} \left(\frac{a+i}{2a} \frac{U_{n,t}}{U_n} \right) - a \left(\left| \left(\frac{U_{n+1}}{U_n} \right)^{\frac{a+i}{2a}} \right|^2 + \left| \left(\frac{U_{n-1}}{U_n} \right)^{\frac{a+i}{2a}} \right|^2 + 2 \right) \right] \left[\left| \left(\frac{U_{n+1}}{U_n} \right)^{\frac{a-i}{2a}} \right|^2 + \left| \left(\frac{U_{n-1}}{U_n} \right)^{\frac{a-i}{2a}} \right|^2 \right] \\ & + \alpha |U_n|^2 \left(\left| \left(\frac{U_{n+1}}{U_n} \right)^{\frac{a+i}{2a}} \right|^2 + \left| \left(\frac{U_{n-1}}{U_n} \right)^{\frac{a+i}{2a}} \right|^2 \right) \left(\left| \left(\frac{U_{n+1}}{U_n} \right)^{\frac{a-i}{2a}} \right|^2 + \left| \left(\frac{U_{n-1}}{U_n} \right)^{\frac{a-i}{2a}} \right|^2 \right) = 0. \end{aligned} \quad (128)$$

In what follows we put $a = 1$ and $\alpha = -8$ for the sake of simplicity. With these special parameters, the semi-discrete equations (119)-(120) are simplified as follows

$$u_{n,t} + (1 + e^{u_n+v_n})(e^{u_{n+1}-u_n} + e^{u_{n-1}-u_n}) - 2 = 0, \quad (129)$$

$$v_{n,t} - (1 + e^{u_n+v_n})(e^{v_{n+1}-v_n} + e^{v_{n-1}-v_n}) + 2 = 0. \quad (130)$$

By the transformation $e^{u_n} \rightarrow u_n$ and $e^{v_n} \rightarrow v_n$, Eqs. (129)-(130) can be expressed in an alternative simpler form

$$u_{n,t} - 2u_n + (1 + u_n v_n)(u_{n+1} + u_{n-1}) = 0, \quad (131)$$

$$v_{n,t} + 2v_n - (1 + u_n v_n)(v_{n+1} + v_{n-1}) = 0. \quad (132)$$

3.3 Soliton solutions of the integrable semi-discrete version of RNLS

In this section, we construct the multi-soliton solutions of the semi-discrete RNLS equation (128) and hence show its integrability. We rewrite Eqs. (116)-(118) with $\delta = a = 1$ in following compact form

$$(D_t - 2 + 2 \cosh(D_n))g_n \bullet f_n = 0, \quad (133)$$

$$(D_t + 2 - 2 \cosh(D_n))h_n \bullet f_n = 0, \quad (134)$$

$$2 \sinh^2\left(\frac{D_n}{2}\right) f_n \bullet f_n - g_n h_n = 0. \quad (135)$$

Similar to the continuous case, we expand f_n, g_n and h_n into series with respect to a small parameter ϵ as follows

$$f_n = 1 + \epsilon^2 f_n^{(2)} + \epsilon^4 f_n^{(4)} + \dots + \epsilon^{2k} f_n^{(2k)} + \dots, \quad (136)$$

$$g_n = \epsilon g_n^{(1)} + \epsilon^3 g_n^{(3)} + \dots + \epsilon^{2k+1} g_n^{(2k+1)} + \dots, \quad (137)$$

$$h_n = \epsilon h_n^{(1)} + \epsilon^3 h_n^{(3)} + \dots + \epsilon^{2k+1} h_n^{(2k+1)} + \dots. \quad (138)$$

We obtain the one-soliton solution

$$g_n = \epsilon \exp(\eta_1), \quad h_n = \epsilon \exp(\eta'_1), \quad f_n = 1 + \frac{\epsilon^2}{4} \text{csch}^2\left(\frac{\beta_1 + \beta'_1}{2}\right) \exp(\eta_1 + \eta'_1), \quad (139)$$

where $\eta_1 = \alpha_1 t + \beta_1 n + \gamma_1$, $\eta'_1 = \alpha'_1 t + \beta'_1 n + \gamma'_1$, and β_1, β'_1 satisfy the dispersion relation

$$\alpha_1 + 4 \sinh^2\left(\frac{\beta_1}{2}\right) = 0, \quad (140)$$

$$\alpha'_1 - 4 \sinh^2\left(\frac{\beta'_1}{2}\right) = 0. \quad (141)$$

Here α_1, γ_1 and α'_1, γ'_1 are arbitrary parameters. The two-soliton solution is presented as follows

$$g_n = \epsilon [\exp(\eta_1) + \exp(\eta_2)] + \epsilon^3 [\varrho_1 \exp(\eta_1 + \eta_2 + \eta'_1) + \varrho_2 \exp(\eta_1 + \eta_2 + \eta'_2)], \quad (142)$$

$$h_n = \epsilon [\exp(\eta'_1) + \exp(\eta'_2)] + \epsilon^3 [\varsigma_1 \exp(\eta'_1 + \eta'_2 + \eta_1) + \varsigma_2 \exp(\eta'_1 + \eta'_2 + \eta_2)], \quad (143)$$

$$\begin{aligned} f_n = 1 + \epsilon^2 & \left[\frac{1}{4} \text{csch}^2\left(\frac{\beta_1 + \beta'_1}{2}\right) \exp(\eta_1 + \eta'_1) + \frac{1}{4} \text{csch}^2\left(\frac{\beta_1 + \beta'_2}{2}\right) \exp(\eta_1 + \eta'_2) \right. \\ & + \frac{1}{4} \text{csch}^2\left(\frac{\beta_2 + \beta'_1}{2}\right) \exp(\eta_2 + \eta'_1) + \frac{1}{4} \text{csch}^2\left(\frac{\beta_2 + \beta'_2}{2}\right) \exp(\eta_2 + \eta'_2) \Big] \\ & + \epsilon^4 \chi [\exp(\eta_1 + \eta_2 + \eta'_1 + \eta'_2)], \end{aligned} \quad (144)$$

where

$$\begin{aligned} \varrho_1 &= \frac{(e^{\beta_1 + \beta'_1} - e^{\beta_2 + \beta'_1})^2}{(e^{\beta_1 + \beta'_1} - 1)^2 (e^{\beta_2 + \beta'_1} - 1)^2}, & \varrho_2 &= \frac{(e^{\beta_1 + \beta'_2} - e^{\beta_2 + \beta'_2})^2}{(e^{\beta_1 + \beta'_2} - 1)^2 (e^{\beta_2 + \beta'_2} - 1)^2}, \\ \varsigma_1 &= \frac{(e^{\beta_1 + \beta'_1} - e^{\beta_1 + \beta'_2})^2}{(e^{\beta_1 + \beta'_1} - 1)^2 (e^{\beta_1 + \beta'_2} - 1)^2}, & \varsigma_2 &= \frac{(e^{\beta_2 + \beta'_1} - e^{\beta_2 + \beta'_2})^2}{(e^{\beta_2 + \beta'_1} - 1)^2 (e^{\beta_2 + \beta'_2} - 1)^2}, \\ \chi &= \frac{(e^{\beta'_1} - e^{\beta'_2})^2 (e^{\beta_1} - e^{\beta_2})^2 e^{\beta_1 + \beta_2 + \beta'_1 + \beta'_2}}{(e^{\beta_2 + \beta'_1} - 1)^2 (e^{\beta_2 + \beta'_2} - 1)^2 (e^{\beta_1 + \beta'_1} - 1)^2 (e^{\beta_1 + \beta'_2} - 1)^2}. \end{aligned} \quad (145)$$

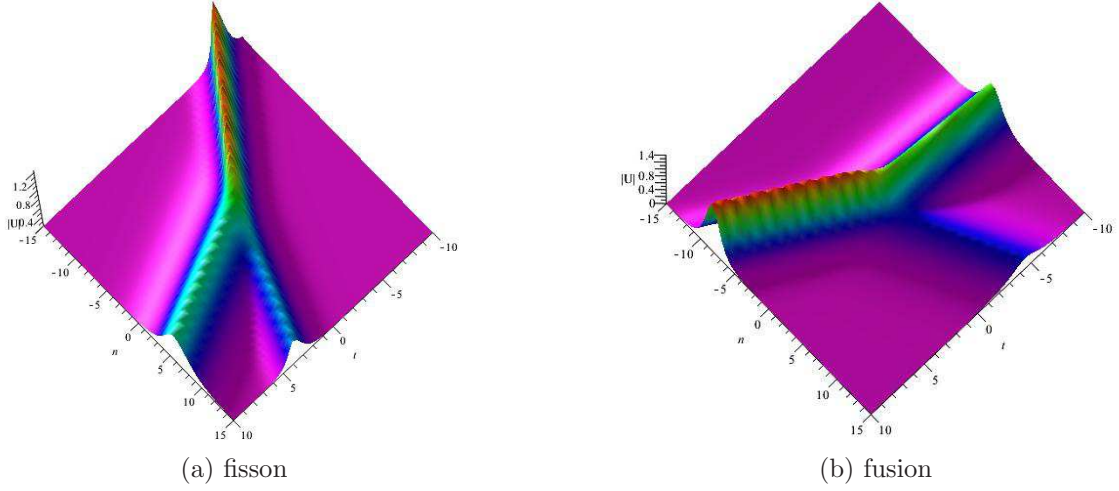


Fig. 3: The resonant interaction of two solitons of discrete RNLS. One soliton splits into two solitons, or, two solitons fuse into one.

The coefficient χ plays a role of classification of soliton interactions. When χ is a finite number and not equal to zero, the regular soliton interaction exists. The resonant interactions of two solitons occur in Eq. (128) in the case of $\chi \rightarrow \infty$ or $\chi \rightarrow 0$. Our discussion is focused on the case of $\chi = 0$ since the other case can be analyzed in a similar way. Parametric conditions is given as follows

$$(e^{\beta'_1} - e^{\beta'_2})(e^{\beta_1} - e^{\beta_2}) = 0. \quad (146)$$

Choosing different factors of (146) as zero, we get two types of soliton resonances, i.e., the fission and fusion. Fig.3 (a) describes one soliton breaks into two solitons with the parameters selected as $\beta_1 = 0.9, \beta_2 = 1.8, \beta'_1 = 0.3, \beta'_2 = 0.3$. Fig.3 (b) shows two solitons fuse into one soliton with the parameters $\beta_1 = 0.8, \beta_2 = 0.8, \beta'_1 = 1.5, \beta'_2 = 0.9$.

Similar as the two-soliton solution (100)-(102) for the continuous equation, we can use the following compact expression for the above two-soliton solution,

$$\begin{aligned} f_n = & 1 + a(1, 1^*) \exp(\eta_1 + \eta'_1) + a(1, 2^*) \exp(\eta_1 + \eta'_2) \\ & + a(2, 1^*) \exp(\eta_2 + \eta'_1) + a(2, 2^*) \exp(\eta_2 + \eta'_2) \\ & + a(1, 2, 1^*, 2^*) \exp(\eta_1 + \eta_2 + \eta'_1 + \eta'_2), \end{aligned} \quad (147)$$

$$\begin{aligned} g_n = & \exp(\eta_1) + \exp(\eta_2) + a(1, 2, 1^*) \exp(\eta_1 + \eta_2 + \eta'_1) \\ & + a(1, 2, 2^*) \exp(\eta_1 + \eta_2 + \eta'_2), \\ h_n = & \exp(\eta'_1) + \exp(\eta'_2) + a(1, 1^*, 2^*) \exp(\eta_1 + \eta'_1 + \eta'_2) \\ & + a(2, 1^*, 2^*) \exp(\eta_2 + \eta'_1 + \eta'_2), \end{aligned} \quad (148)$$

where the coefficients are defined by

$$a(i, j) = 4 \sinh^2\left(\frac{\beta_i - \beta_j}{2}\right), \quad (149)$$

$$a(i, j^*) = \frac{1}{4} \text{csch}^2\left(\frac{\beta_i + \beta'_j}{2}\right), \quad (150)$$

$$a(i^*, j^*) = 4 \sinh^2\left(\frac{\beta'_i - \beta'_j}{2}\right). \quad (151)$$

The exact N -soliton solution of eqs. (133)-(135) is presented in the form

$$f_n = \sum_{\mu=0,1}^{(e)} \exp \left[\sum_{j=1}^N \mu_j \eta_j + \sum_{j=N+1}^{2N} \mu_j \eta'_{j-N} + \sum_{1 \leq k < l}^{2N} \mu_k \mu_l A_{kl} \right], \quad (152)$$

$$g_n = \sum_{\nu=0,1}^{(o)} \exp \left[\sum_{j=1}^N \nu_j \eta_j + \sum_{j=N+1}^{2N} \nu_j \eta'_{j-N} + \sum_{1 \leq k < l}^{2N} \nu_k \nu_l A_{kl} \right], \quad (153)$$

$$h_n = \sum_{\lambda=0,1}^{(o)} \exp \left[\sum_{j=1}^N \lambda_j \eta_j + \sum_{j=N+1}^{2N} \lambda_j \eta'_{j-N} + \sum_{1 \leq k < l}^{2N} \lambda_k \lambda_l A_{kl} \right], \quad (154)$$

where

$$\eta_j = \alpha_j t + \beta_j n + \gamma_j, \quad \alpha_j = -4 \sinh^2 \frac{\beta_j}{2}, \quad j = 1, 2, \dots, N, \quad (155)$$

$$\eta'_j = \alpha'_j t + \beta'_j n + \gamma'_j, \quad \alpha'_j = 4 \sinh^2 \frac{\beta'_j}{2}, \quad j = 1, 2, \dots, N, \quad (156)$$

$$\exp(A_{kl}) = 4 \sinh^2 \left(\frac{\beta_k - \beta_l}{2} \right), \quad k < l = 2, 3, \dots, N, \quad (157)$$

$$\exp(A_{k,N+l}) = \frac{1}{4} \operatorname{csch}^2 \left(\frac{\beta_k + \beta'_l}{2} \right), \quad k, l = 1, 2, \dots, N, \quad (158)$$

$$\exp(A_{N+k,N+l}) = 4 \sinh^2 \left(\frac{\beta'_k - \beta'_l}{2} \right), \quad k < l = 2, 3, \dots, N. \quad (159)$$

The summations $\sum_{\mu=0,1}^{(e)}$, $\sum_{\nu=0,1}^{(o)}$ and $\sum_{\lambda=0,1}^{(o)}$ satisfy the condition (26)-(28) respectively.

4 Conclusions

To summarize, we presented here one semi-discrete integrable version for the (2+1)-dimensional modified HF equation and one semi-discrete version for the resonant NLS equation. N -soliton solution to the both discrete systems are given in standard form. Interaction properties of two-soliton solutions are analyzed. The regular, intermediate-state and resonant soliton interactions of the semi-discrete systems are discussed. Those analysis on the resonant interactions might have the applications in optical communication systems. Fully discrete integrable version for the two equations is under consideration.

Acknowledgements

The work was supported in part by the Research Grants Council of the Hong Kong Special Administrative Region, China (GRF600806). and National Natural Science Foundation of China (Grant no.11371251).

References

- [1] Y. B. Suris: solv-int/9709005 v4 (1997).
- [2] W. Heisenberg, Z. Phys. 49 (1928) 619.
- [3] H. J. Mikeska and M. Steiner, Adv. Phys. 40 (1991) 191.
- [4] Q. Ding, W. Lin, J. Math. Phys. 49 (2008) 093501.
- [5] S.C. Anco, R. Myrzakulov, J. Geom. Phys. 60 (2010) 1576.
- [6] X.J. Duan, M. Deng, W.Z. Zhao, K. Wu, J. Phys. A 40 (2007) 3831.
- [7] B.B. Thomas, K.K. Victor, K.T. Crepin, J. Phys. A 41 (2008) 135208.

- [8] K.K. Victor, B.B. Thomas, T.C. Kofane, *Phys. Rev. E* **79** (2009) 056605.
- [9] A.J. Koch, H. Meinhardt, *Rev. Mod. Phys.* **66** (1994) 1481.
- [10] G.Q. Meng, etc, *Computers and Mathematics with Applications*, **66** 2559.
- [11] O.K. Pashaev, J.H. Lee, *Modern Phys. Lett. A* **17** (2002) 1601.
- [12] J.H. Lee, O.K. Pashaev, C. Rogers, W.K. Schief, *J. Plasma Phys.* **73** (2007) 257.
- [13] J.H. Lee, O.K. Pashaev, *Theor. Math. Phys.* **152** (2007) 991.
- [14] O.K. Pashaev, J.H. Lee, C. Rogers, *J. Phys. A: Math. Theor.* **41** (2008) 452001
- [15] O.K. Pashaev, J.H. Lee, *ANZIAM J.* **44** (2002) 73.
- [16] Min Li, Jing-Hua Xiao, Tian-Zhong Yan, Bo Tian, *Nonlinear Analysis: Real World Applications* **14** (2013) 1669.
- [17] R. Hirota, *Direct method in soliton theory (In English)*, (Edited and Translated by A.Nagai, J.Nimmo and C.Gilson, Cambridge University Press, 2004).
- [18] R. Hirota, *J. Math. Phys.* **14** (1973), 805
- [19] R. Hirota, Discretization of coupled soliton equations, *Bilinear Integrable Systems: From Classical to Quantum, Continuous to Discrete*, NATO Sci. Ser. II Math. Phys. Chem., **201**, Springer, Dordrecht, (2006) 113.
- [20] L.Vinet and G.F. Yu, *J.Nonlinear.Math.Phys.* **20** (2013), 106.